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Radiative solutions for a rapidly rotating magnetised sphere

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Abstract. Maxwell's equations in flat spacetime are solved for an isolated uniformly rotating magnetised sphere that can sustain convective and vorticity currents in the infinite conductivity limit. The study is motivated by the problem of accounting for the radiation from rapidly spinning compact stellar objects. Two types of magnetic interiors are considered to determine how the resultant radiation field departs from traditional point dipole magnetisation models. The torque reaction produced by the emitted radiation is calculated and the result compared with simpler models that have featured in the dynamics of pulsars.

1. Introduction

The electrodynamic properties of moving media are relevant to a number of problems in contemporary physics. This paper investigates the electromagnetic field generated by a rapidly rotating, rigid, magnetised sphere *in vacuo*. Such a problem has been encountered by various authors in a number of different scenarios [1, 2]. In particular the description of radiation from compact objects such as pulsars and magnetic stars has relied fundamentally on such field configurations. Many authors in this context have developed early work by Deutsch [3] (see also [4]) with a number of refinements to a basically magnetic dipole model. The essential programme in these papers is the solution of Maxwell's equation in an inertial frame by postulating some co-rotating magnetic field in the interior of the star [5]. Then if the conductivity of the medium is effectively infinite one can immediately deduce the interior electric field from an assumed rigid rotation rate. By continuity the tangential electric and normal interior magnetic induction fields with respect to the surface then determine the structure of the Maxwell solutions outside the star.

An alternative procedure is to investigate the Einstein-Maxwell system for modifications to the Kerr metric. This is relevant for those situations in which the electromagnetic field is influenced by an intense gravitational field such as that around a rotating magnetic black hole. A perturbative approach to this problem has been pioneered by Teukolsky [6] and Teukolsky and Press [7]. In this approach the field boundary conditions on various horizons play an important role in constructing solutions. In such problems one is rarely concerned with interior stellar solutions and the subsequent matching problems for electromagnetic fields at magnetisation boundaries. The complexity of coupled Einstein-Maxwell problems for realistic black holes has led to the *membrane paradigm* [8] for dealing with boundary data. We shall not in this paper be concerned with the influence of gravitation on the Maxwell equations and will concentrate on exact relativistic solutions in Minkowski spacetime.

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Even with a non-dynamical (flat) gravitational background the procedures outlined above leave great freedom in the construction of realistic models. To our knowledge most models to date postulate in this context an interior magnetic field corresponding to a *point* magnetic dipole situated at some location within an infinitely conducting sphere. With a central dipole aligned along the axis of rotation the external field consists of a static magnetic dipole field together with a static electric quadrupole field. In the non-aligned case [9] the near field includes magnetic quadrupole terms and one finds electromagnetic radiation from the sphere. In the non-relativistic limit such solutions can be simulated by rotating a permanent uniform magnetisation within the sphere. Since a static uniformly magnetised sphere behaves as a point magnetic dipole at the centre of a sphere with respect to the external fields one might expect point dipole models to always be simulations of simple magnetised interiors, even in the case of rapidly rotating spheres. It is this issue that we address in the following: we seek exact solutions in which the form of the interior co-rotating magnetisation is a self-consistent source. The form taken by the magnetisation will depend on the physics of extended matter under extreme conditions. Such conditions are unlikely to be simulated in the laboratory or probed directly by astrophysical observation. In such circumstances one can but attempt to reconcile observational data with theoretical models. In this paper we present two solutions corresponding to two different types of magnetisation. In both cases we admit the existence of a purely convective current in the interior of the sphere. Such a current is demanded by the phenomenon of unipolar induction and has been studied in [10]. However, if the extended matter is magnetically 'pliable' so that the local magnetisation is deformable then a solution can be found without generating additional currents. On the other hand if the matter is magnetically 'stiff', a different type of interior solution may be found in which additional electric vorticity currents flow. These solutions produce markedly different radiation patterns in general. The field exterior to the sphere is essentially determined by matching to a component of the interior Maxwell field F. The different types of magnetisation determine different surface currents and charges on the interface between the sphere and the vacuum. It is this latter characteristic that may be of relevance in more complex models in which the sphere rotates in a plasma and is one of the motivations for our study of solutions describing extended magnetic matter. Once one leaves the simple point dipole model, generalisations become somewhat ad hoc unless they can be related to particular interior solutions to the relativistic Maxwell equations for an accelerating source.

2. Interior fields

Our formulation of this problem follows that of [10]. Thus we use the language of exterior forms. A notable difference from earlier treatments of Maxwell equations in this language [11] is that we work directly with decompositions of the Maxwell 2-form and the associated magnetisation source form rather than a potential. The construction of our solutions is facilitated by the choice of a convenient Frenet frame adapted to the problem. In [10] the stationary fields generated by a rapidly rotating magnetised sphere were derived for an axially symmetric configuration. It is this symmetry that we now relax, allowing for the possibility of electromagnetic radiation. As before the rotating sphere is modelled by a time-like unit vector field V with compact support on a domain I of Minkowski spacetime M. This interior of the sphere has a history defined by a bounding hypersurface. Such a hypersurface may be taken as the zero of a real function f on spacetime that partitions the manifold M into the domain I ($f<0$, the interior history) and II ($f>0$, the exterior domain). For a solid sphere, I is contractable and the history of the centre of the sphere is taken to be an inertial integral curve of V.

Denote by g the Minkowski metric tensor field on M and by $*$ its associated Hodge map on differential forms. Let $F^{\rm I}$ be the Maxwell field 2-form on domain I and $\Pi^{\rm I}$ the polarisation 2-form describing the permanent magnetisation of the sphere. Let $j^{\rm I}$ denote the current 3-form describing all Maxwell sources not included in $\Pi^{\rm I}$. Then Maxwell's equations for the fields in the sphere may be written

$$\mathrm{d}F^{\rm I}=0 \tag{2.1}$$

$$\mathrm{d} * G^{\rm I}=j^{\rm I} \tag{2.2}$$

where $G^{\rm I}=\varepsilon_0 F^{\rm I}+\Pi^{\rm I}$ in terms of the permittivity ε_0 of free spacetime. (Throughout we use units in which $c=1$ so $\varepsilon_0\mu_0=1$.) If domain I is an Ohmic conductor with constant scalar conductivity σ then $*j^{\rm I}$ contains a contribution $-\sigma i_V F^{\rm I}$. We look for field configurations for which the only additional allowed currents in I comprise a convective term $-\rho\tilde{V}$ where ρ is a scalar function (the volume charge density in the co-rotating frame of the sphere) to be determined, together with a possible vorticity current depending on $\mathrm{d}\tilde{V}$. Here and in what follows a tilde over a vector denotes the form associated with it by the metric; i.e. $\tilde{V}(X)=g(V,X)$ for all vector fields X.

It is worth emphasising that the choice of admissible currents $(j, \mathrm{d} * \Pi)^{\rm I}$ in I properly characterises the class of models under consideration. In the following we consider the possibility of a vorticity current $\lambda\,\mathrm{d}\tilde{V}\wedge\tilde{V}$ where λ is a scalar on I.

The magnetic properties of the sphere are encoded into the space-like 1-form $\boldsymbol{m}_V$ on I satisfying $i_V\boldsymbol{m}_V=0$. This produces a co-moving magnetic induction field 1-form $\boldsymbol{b}_V$ satisfying $i_V\boldsymbol{b}_V=0$. We assume that the Ohmic conductivity of the sphere is effectively infinite so that

$$i_V F^{\rm I}=0. \tag{2.3}$$

Thus we write

$$\Pi^{\rm I}=-*(\boldsymbol{m}_V\wedge\tilde{V}) \tag{2.4}$$

$$F^{\rm I}=*(\boldsymbol{b}_V\wedge\tilde{V})\equiv i_V * \boldsymbol{b}_V. \tag{2.5}$$

To proceed we introduce a local spherical polar coordinate system (t, r, θ, ϕ) in which the metric takes the form

$$g=-\mathrm{d}t\otimes\mathrm{d}t+\mathrm{d}r\otimes\mathrm{d}r+r^2\,\mathrm{d}\theta\otimes\mathrm{d}\theta+r^2\sin^2\theta\,\mathrm{d}\phi\otimes\mathrm{d}\phi. \tag{2.6}$$

In these coordinates the hypersurface $f=0$ has $f=r-a$ where the constant a may be identified with the radius of the sphere. If the constant ω denotes the angular velocity of the sphere about a fixed direction we have

$$V=\gamma(\partial_t+\omega\partial_\phi) \tag{2.7}$$

where

$$\gamma(r,\theta)=(1-\omega^2r^2\sin^2\theta)^{-1/2}. \tag{2.8}$$

The vector field V will remain time-like in I provided ω is constrained to ensure that γ remains bounded. Since $i_V F^{\rm I}=0$ and $\mathrm{d}F^{\rm I}=0$ it follows that $\mathcal{L}_V F^{\rm I}=0$ in terms of the Lie derivative. The above conditions can be implemented by taking the forms $\boldsymbol{m}_V$ and $\boldsymbol{b}_V$ to depend on functions of $(\omega t-\phi)$. We further demand that in the limit $\omega=0$ our solutions reduce in an inertial frame to the standard magnetostatic solutions for a sphere with a *uniform constant* magnetisation, having a non-zero angle with the rotation axis. The above conditions certainly include those imposed by other authors in their analyses and we therefore expect to make contact with the solutions of Deutsch in some limit or approximation. What is essentially new in the following is an investigation of Maxwell's equations for the interior of the rotating sphere with a self-consistent magnetisation and its subsequent effect on the exterior radiation modes.

To determine $\boldsymbol{b}_V$ and $\boldsymbol{m}_V$ we shall refer them to a field of Frenet frames $\{N_i\}$ defined by V. When restricted to any integral curve C of V (the history of an element of a sphere) the set $\{N_i\}|_C$ provides a well defined g-orthonormal Frenet tetrad along C. If ∇ denotes the Levi-Civita connection such a frame is defined to satisfy

$$\nabla_{N_0}N_0|_C=\lambda_0\kappa_1N_1 \tag{2.9}$$

$$\nabla_{N_0}N_1|_C=-\lambda_0\kappa_1N_0+\lambda_2\kappa_2N_2 \tag{2.10}$$

$$\nabla_{N_0}N_2|_C=-\lambda_1\kappa_2N_1+\lambda_3\kappa_3N_3 \tag{2.11}$$

$$\nabla_{N_0}N_3|_C=-\lambda_2\kappa_3N_2 \tag{2.12}$$

where $g(N_i, N_i)=\lambda_i=\pm1$ and κ_i are scalars (the curvatures of C). We shall choose N_0 time-like. These equations have as solutions (naturally extended to a field):

$$N_0=V=\gamma(X_0+\omega r\sin\theta\, X_3) \tag{2.13}$$

$$N_1=\sin\theta\, X_1+\cos\theta\, X_2 \tag{2.14}$$

$$N_2=\gamma(\omega r\sin\theta\, X_0+X_3) \tag{2.15}$$

$$N_3\equiv\partial_z=\cos\theta\, X_1-\sin\theta\, X_2 \tag{2.16}$$

in terms of the orthobasis $(X_0=\partial_t,\ X_1=\partial_r,\ X_2=(1/r)\,\partial_\theta,\ X_3=1/(r\sin\theta)\,\partial_\phi)$. The curvatures of C are $(\kappa_1=-\gamma^2\omega^2r\sin\theta,\ \kappa_2=\omega\gamma^2,\ \kappa_3=0)$. We verify that if $e^0=\mathrm{d}t$, $(e^1=\mathrm{d}r,\ e^2=r\,\mathrm{d}\theta,\ e^3=r\sin\theta\,\mathrm{d}\phi)$ then

$$*1\equiv e^0\wedge e^1\wedge e^2\wedge e^3=-\tilde{N}_0\wedge\tilde{N}_1\wedge\tilde{N}_2\wedge\tilde{N}_3.$$

In such a frame we may expand

$$\boldsymbol{b}_V=\sum_{i=1}^{3}\alpha_i\tilde{N}_i \tag{2.17}$$

$$\boldsymbol{m}_V=\sum_{i=1}^{3}\beta_i\tilde{N}_i \tag{2.18}$$

in terms of six functions $\{\alpha_i, \beta_i\}$ on I.

We substitute the expansions (2.17) and (2.18) into (2.1) and (2.2) with a choice of sources $(j, \mathrm{d}*\Pi)$ of the form discussed above. The symmetry conditions, together with the condition that we desire the magnetisation of the non-rotating sphere to be uniform and directed at a fixed angle to the rotation axis, enable us to select a solution with

$$\alpha_1=-\frac{1}{\gamma}[b_1\cos(\omega t-\phi)-b_2\sin(\omega t-\phi)] \tag{2.19}$$

$$\alpha_2=-[b_1\sin(\omega t-\phi)+b_2\cos(\omega t-\phi)] \tag{2.20}$$

$$\alpha_3=\gamma b_3. \tag{2.21}$$

Hence we have a closed 2-form

$$F^{1a} = b_1\left\{\sin(\omega t-\phi)\tilde{N}_1\wedge\tilde{N}_3 - \frac{\cos(\omega t-\phi)}{\gamma}\tilde{N}_2\wedge\tilde{N}_3\right\}$$

$$+ b_2\left\{\frac{\sin(\omega t-\phi)}{\gamma}\tilde{N}_2\wedge\tilde{N}_3 - \cos(\omega t-\phi)\tilde{N}_1\wedge\tilde{N}_3\right\} - b_3\gamma\tilde{N}_1\wedge\tilde{N}_2 \qquad (2.22)$$

with (b_1, b_2, b_3) constants to be determined.

(The fact that this form is closed follows readily with the aid of the equations:

$$\mathrm{d}\tilde{N}_0 = \frac{\gamma^2-1}{r\sin\theta}\tilde{N}_0\wedge\tilde{N}_1 + 2\gamma^2\omega\tilde{N}_1\wedge\tilde{N}_2 \qquad (2.23)$$

$$\mathrm{d}\tilde{N}_1 = 0 \qquad (2.24)$$

$$\mathrm{d}\tilde{N}_2 = \frac{\gamma^2}{r\sin\theta}\tilde{N}_1\wedge\tilde{N}_2 \qquad (2.25)$$

$$\mathrm{d}\tilde{N}_3 = 0 \qquad (2.26)$$

and confirms the utility of the Frenet basis here.) In terms of the inertial co-frame:

$$F^{1a} = (-b_1\cos\alpha + b_2\sin\alpha)((\omega r\sin^2\theta - \omega r\sin\theta\cos\theta)e^0\wedge e^1$$
$$-\cos\theta\, e^1\wedge e^3 + \sin\theta\, e^2\wedge e^3) + (b_1\sin\alpha + b_2\cos\alpha)(e^2\wedge e^1)$$
$$-\gamma^2 b_3(\omega r\sin^2\theta\, e^0\wedge e^1 + \omega r\sin\theta\cos\theta\, e^0\wedge e^2$$
$$+\cos\theta\, e^2\wedge e^3 + \sin\theta\, e^1\wedge e^3) \qquad (2.27)$$

where $\alpha = (\omega t - \phi)$. The magnetisation form Π^{1a} that enters into the Maxwell equation

$$\mathrm{d} * G^1 = -*\rho\tilde{V} + \lambda\,\mathrm{d}\tilde{V}\wedge\tilde{V} \qquad (2.28)$$

is proportional to F^{1a}:

$$\Pi^{1a} = (m_1\cos\alpha - m_2\sin\alpha)((\omega r\sin^2\theta - \omega r\sin\theta\cos\theta)e^0\wedge e^1$$
$$-\cos\theta\, e^1\wedge e^3 + \sin\theta\, e^2\wedge e^3) + (m_1\sin\alpha + m_2\cos\alpha)(e^2\wedge e^1)$$
$$-\gamma^2 m_3(\omega r\sin^2\theta\, e^0\wedge e^1 + \omega r\sin\theta\cos\theta\, e^0\wedge e^2$$
$$+\cos\theta\, e^2\wedge e^3 + \sin\theta\, e^1\wedge e^3) \qquad (2.29)$$

in terms of some given constants m_1, m_2, m_3. The magnetisation is thus specified by

$$\beta_1 = -\frac{1}{\gamma}[m_1\cos(\omega t-\phi) - m_2\sin(\omega t-\phi)] \qquad (2.30)$$

$$\beta_2 = -[m_1\sin(\omega t-\phi) + m_2\cos(\omega t-\phi)] \qquad (2.31)$$

$$\beta_3 = \gamma m_3. \qquad (2.32)$$

We then have a solution with

$$\rho = -2\gamma^3\omega(\varepsilon_0 b_3 - m_3) \qquad (2.33)$$

$$\lambda = -\frac{\omega r\sin\theta}{2\gamma^2}[(\varepsilon_0 b_1 - m_1)\sin\alpha + (\varepsilon_0 b_2 - m_2)\cos\alpha]. \qquad (2.34)$$

Such a solution corresponds to what we have termed a stiff magnetisation in the introduction. From the expression for Π^{1_a} we have

$$G^{1_a} = (-c_1 \cos\alpha + c_2 \sin\alpha)((\omega r \sin^2\theta - \omega r \sin\theta\cos\theta)e^0 \wedge e^1 - \cos\theta\, e^1 \wedge e^3 + \sin\theta\, e^2 \wedge e^3) + (c_1 \sin\alpha + c_2 \cos\alpha)(e^2 \wedge e^1) - \gamma^2 c_3(\omega r \sin^2\theta\, e^0 \wedge e^1 + \omega r \sin\theta\cos\theta\, e^0 \wedge e^2 + \cos\theta\, e^2 \wedge e^3 + \sin\theta\, e^1 \wedge e^3) \tag{2.35}$$

where $c_i = \varepsilon_0 b_i - m_i$ for $i = 1, 2, 3$.

It is of interest now to look for a different solution corresponding to a magnetisation that generates a purely convective current source, i.e. one with $\lambda = 0$. This can be achieved if we modify the constitutive relation between F^1 and Π^1, allowing the proportionality to vary over the interior of the sphere in a way consistent with Maxwell's equations. However we shall impose the conditions

$$\partial_t(g(\widetilde{m_V}, \widetilde{m_V})) = 0 \tag{2.36}$$

$$\partial_\phi(g(\widetilde{m_V}, \widetilde{m_V})) = 0 \tag{2.37}$$

so that the magnetisation simply adjusts its local direction uniformly in each concentric shell about the origin, in response to the rotation. With the imposition of this condition the $\{\alpha_i\}$ follow as before:

$$\alpha_1 = \mathscr{P}(\gamma)(b_1 \cos(\omega t - \phi) - b_2 \sin(\omega t - \phi)) \tag{2.38}$$

$$\alpha_2 = \mathscr{P}(\gamma)(b_1 \sin(\omega t - \phi) + b_2 \cos(\omega t - \phi)) \tag{2.39}$$

$$\alpha_3 = \gamma b_3 \tag{2.40}$$

$$\mathscr{P}(\gamma) = \frac{2}{e}\frac{\exp(1/\gamma)}{(1+\gamma)} \tag{2.41}$$

with $\mathscr{P}$ normalised so that $\mathscr{P}(1) = 1$, and $\{b_1, b_2, b_3\}$ constants to be determined.

Similarly the equation (2.28) with $\lambda = 0$ is solved with

$$\beta_1 = \mathscr{G}(\gamma)(m_1 \cos(\omega t - \phi) - m_2 \sin(\omega t - \phi)) \tag{2.42}$$

$$\beta_2 = \mathscr{G}(\gamma)(m_1 \sin(\omega t - \phi) + m_2 \cos(\omega t - \phi)) \tag{2.43}$$

$$\beta_3 = \gamma m_3 \tag{2.44}$$

and

$$\mathscr{G}(\gamma) = \frac{2}{e}(\gamma + s(1-\gamma))\frac{\exp(1/\gamma)}{1+\gamma} \tag{2.45}$$

normalised to $\mathscr{G}(1) = 1$. Such a solution, denoted henceforth as solution *b*, corresponds to what we have termed a pliable magnetisation in the introduction. For consistency we must have the constant $s = \varepsilon_0 b_1/m_1 = \varepsilon_0 b_2/m_2$. The constants $\{b_i\}$ in both types of solution will be fixed by our boundary conditions in terms of the constants $\{m_i\}$ that specify the overall strength of the static magnetisation. Besides fixing the structure of the function $\mathscr{G}$, equation (2.28) also determines the convective current $-\rho\tilde{V}$. Thus the co-moving volume charge density within the sphere induced by the rotation is given by

$$\rho = -2\gamma^3\omega(\varepsilon_0 b_3 - m_3).$$

Both stiff and pliable magnetic interiors generate the same convective current, determined essentially by the component of magnetisation in the direction of rotation.

In the inertial frame determined by the observer field $U = \partial_t$ the volume charge density follows from

$$-\rho * \tilde{V} = -\rho_U * \tilde{U} + J_U \wedge \tilde{U}. \tag{2.46}$$

Hence the inertial charge density for both solutions is

$$-\rho_U = -\rho \tilde{V}(\partial_t) = -2(\varepsilon_0 b_3 - m_3)\gamma^4 \omega \tag{2.47}$$

and is accompanied by an azimuthal 3-current 2-form

$$J_U = \rho_U \omega r^2 \sin\theta \, \mathrm{d}r \wedge \mathrm{d}\theta. \tag{2.48}$$

The total free volume charge Q_{vol} inside the sphere is readily computed from the formula

$$Q_{\text{vol}} = \int_{S_a^2} * G^1 = -\frac{4\pi}{\omega}(\varepsilon b_3 - m_3)\left\{-\omega a + \frac{\sin^{-1}\omega a}{\sqrt{1-\omega^2 a^2}}\right\} \tag{2.49}$$

and this vanishes when $\omega = 0$.

For future reference we record that in the pliable magnetisation case:

$$\begin{aligned} F^{\mathrm{I}_b} = {} & \gamma\mathscr{P}(\gamma)(-b_1\cos\alpha + b_2\sin\alpha)((\omega r\sin^2\theta - \omega r\sin\theta\cos\theta)e^0\wedge e^1 \\ & -\cos\theta\, e^1\wedge e^3 + \sin\theta\, e^2\wedge e^3) + \mathscr{P}(\gamma)(b_1\sin\alpha + b_2\cos\alpha)(e^2\wedge e^1) \\ & -\gamma^2 b_3(\omega r\sin^2\theta\, e^0\wedge e^1 + \omega r\sin\theta\cos\theta\, e^0\wedge e^2 \\ & +\cos\theta\, e^2\wedge e^3 + \sin\theta\, e^1\wedge e^3) \end{aligned} \tag{2.50}$$

$$\begin{aligned} \Pi^{\mathrm{I}_b} = {} & \gamma\mathscr{G}(\gamma)(m_1\cos\alpha - m_2\sin\alpha)((\omega r\sin^2\theta - \omega r\sin\theta\cos\theta)e^0\wedge e^1 \\ & -\cos\theta\, e^1\wedge e^3 + \sin\theta\, e^2\wedge e^3) + \mathscr{G}(\gamma)(m_1\sin\alpha + m_2\cos\alpha)(e^2\wedge e^1) \\ & -\gamma^2 m_3(\omega r\sin^2\theta\, e^0\wedge e^1 + \omega r\sin\theta\cos\theta\, e^0\wedge e^2 \\ & +\cos\theta\, e^2\wedge e^3 + \sin\theta\, e^1\wedge e^3) \end{aligned} \tag{2.51}$$

$$\begin{aligned} G^{\mathrm{I}_b} = {} & \gamma(-\mathscr{H}_1(\gamma)\cos\alpha + \mathscr{H}_2(\gamma)\sin\alpha)((\omega r\sin^2\theta - \omega r\sin\theta\cos\theta)e^0\wedge e^1 \\ & -\cos\theta\, e^1\wedge e^3 + \sin\theta\, e^2\wedge e^3) + (\mathscr{H}_1(\gamma)\sin\alpha + \mathscr{H}_2(\gamma)\cos\alpha)(e^2\wedge e^1) \\ & -\gamma^2(\varepsilon_0 b_3 - m_3)(\omega r\sin^2\theta\, e^0\wedge e^1 + \omega r\sin\theta\cos\theta\, e^0\wedge e^2 \\ & +\cos\theta\, e^2\wedge e^3 + \sin\theta\, e^1\wedge e^3). \end{aligned} \tag{2.52}$$

In these expressions

$$\mathscr{H}(\gamma) = \mathscr{H}(\gamma)_1 - \mathrm{i}\mathscr{H}(\gamma)_2$$

$$\mathscr{H}(\gamma)_i = \varepsilon_0 b_i \mathscr{P}(\gamma) - m_i \mathscr{G}(\gamma) \qquad i = 1, 2.$$

In order to fix the constants $\{b_i\}$ in both solutions we must compute the vacuum fields outside the sphere.

3. Exterior fields

In the domain $r > a$ we decompose $F^{\rm II}$ with respect to $U = \partial_t$:

$$F^{\rm II} = \boldsymbol{e}^{\rm II} \wedge \tilde{U} + *(\boldsymbol{b}^{\rm II} \wedge \tilde{U}) \tag{3.1}$$

$$= -\boldsymbol{e}^{\rm II} \wedge \mathrm{d}t - \hat{*}\, \boldsymbol{b}^{\rm II} \tag{3.2}$$

where $* 1 = \mathrm{d}t \wedge \hat{*} 1$ and $i_U \boldsymbol{e}^{\rm II} = i_U \boldsymbol{b}^{\rm II} = 0$. The real electric and magnetic 1-form fields satisfy the inertial vacuum Maxwell equations

$$\mathrm{d}\boldsymbol{e}^{\rm II} = -\hat{*}\, \mathscr{L}_{\partial_t} \boldsymbol{b}^{\rm II} \tag{3.3}$$

$$\mathrm{d}\, \hat{*}\, \boldsymbol{e}^{\rm II} = 0 \tag{3.4}$$

$$\mathrm{d}\boldsymbol{b}^{\rm II} = \hat{*}\, \mathscr{L}_{\partial_t} \boldsymbol{e}^{\rm II} \tag{3.5}$$

$$\mathrm{d}\, \hat{*}\, \boldsymbol{b}^{\rm II} = 0. \tag{3.6}$$

Guided by the structure of $F^{\rm I}$ we write

$$\boldsymbol{e}^{\rm II} = \sum_{n=-\infty}^{\infty} \boldsymbol{e}_n \,\mathrm{e}^{-\mathrm{i}k_n t} - \mathrm{d}\Phi_{\rm E}^{\rm II} \tag{3.7}$$

$$\boldsymbol{b}^{\rm II} = \sum_{n=-\infty}^{\infty} \boldsymbol{b}_n \,\mathrm{e}^{-\mathrm{i}k_n t} - \mathrm{d}\Phi_{\rm M}^{\rm II} \tag{3.8}$$

where $k_n = n\omega$ with $n \neq 0$ and $\boldsymbol{e}_n$, $\boldsymbol{b}_n$, $\mathrm{d}\Phi_{\rm E}^{\rm II}$, $\mathrm{d}\Phi_{\rm M}^{\rm II}$ are time-independent. Henceforth all sums over n will be understood to exclude the term with $n = 0$. The electric and magnetic scalar potentials on II satisfy

$$\partial_\phi \Phi_{\rm E}^{\rm II} = 0 \tag{3.9}$$

$$\partial_\phi \Phi_{\rm M}^{\rm II} = 0. \tag{3.10}$$

Since $\Phi_{\rm E}$ and $\Phi_{\rm M}$ are also independent of time they must be harmonic on II and may be analysed into Legendre polynomials with standard radial amplitudes. From (3.3)–(3.6) it follows that the Fourier modes must satisfy

$$\mathrm{d}\, \hat{*}\, \mathrm{d}\boldsymbol{e}_n = k_n^2\, \hat{*}\, \boldsymbol{e}_n \tag{3.11}$$

$$\mathrm{d}\, \hat{*}\, \mathrm{d}\boldsymbol{b}_n = k_n^2\, \hat{*}\, \boldsymbol{b}_n \tag{3.12}$$

$$\boldsymbol{e}_n = \frac{\mathrm{i}}{k_n}\, \hat{*}\, \mathrm{d}\boldsymbol{b}_n \tag{3.13}$$

$$\boldsymbol{b}_n = -\frac{\mathrm{i}}{k_n}\, \hat{*}\, \mathrm{d}\boldsymbol{e}_n \tag{3.14}$$

where $\bar{\boldsymbol{e}}_n = \boldsymbol{e}_{-n}$ and $\bar{\boldsymbol{b}}_n = \boldsymbol{b}_{-n}$.

It is customary to analyse these equations in a spherical polar chart using vector spherical harmonics. Such vector harmonics are constructed from a 0-form solution to the Helmholtz equation and have been extensively studied [12]. We shall simply observe here that such a mode analysis relies fundamentally on the fact that if a 1-form α on a three-dimensional manifold satisfies the equations

$$\delta\alpha = 0 \tag{3.15}$$

$$\delta\, \mathrm{d}\alpha = -\omega^2 \alpha \tag{3.16}$$

where $\delta \equiv -*^{-1}\mathrm{d} * \alpha$, then $i_X\alpha$ also satisfies the 0-form equation

$$\delta\, \mathrm{d}(i_X\alpha) = -\omega^2(i_X\alpha) \tag{3.17}$$

where X is any homothetic conformal Killing vector field:

$$\mathscr{L}_X g = 2\Lambda g \tag{3.18}$$

for some constant Λ. In R^3 with standard spherical polar coordinates, $X = r\partial_r$ is such a vector field. We write

$$e^{\text{II}} + \mathrm{d}\Phi_{\text{E}}^{\text{II}} \equiv e = \sum_{n=-\infty}^{\infty} e_{\text{TM}}^n + e_{\text{TE}}^n \tag{3.19}$$

$$b^{\text{II}} + \mathrm{d}\Phi_{\text{M}}^{\text{II}} \equiv b = \sum_{n=-\infty}^{\infty} b_{\text{TM}}^n + b_{\text{TE}}^n \tag{3.20}$$

where $i_{\partial_r} e_{\text{TE}}^n = 0$ and $i_{\partial_r} b_{\text{TM}}^n = 0$. Note that $r i_{\partial_r} b_{\text{TM}}^n$ and $r i_{\partial_r} e_{\text{TM}}^n$ may be expanded in terms of the 0-form spherical harmonics $Y_{lm}(\theta, \phi)$ with radial functions proportional to solutions of the spherical Bessel equation. For radiative modes at large r we naturally adopt the spherical Hankel functions $h_l^{(1)}(k_n r)$ so that $h_l^{(1)}(k_n r)\, \mathrm{e}^{-\mathrm{i}k_n t}$ (and its complex conjugate) represent outgoing radial waves. By performing a $2+1$ split of the forms e and b with respect to the radial field ∂_r such that

$$\hat{*}1 = \mathrm{d}r \wedge \# 1 \tag{3.21}$$

we deduce the expansions

$$e^{\text{II}} = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \left(\frac{\mathrm{i}}{k_n} a_{\text{TM}}^n(l, m)\, \hat{*}\, \mathrm{d}(h_{ln} X_{lm}) + a_{\text{TE}}^n(l, m) h_{ln} X_{lm} \right) \mathrm{e}^{-\mathrm{i}k_n t} - \mathrm{d}\Phi_{\text{E}}^{\text{II}} \tag{3.22}$$

$$b^{\text{II}} = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \left(a_{\text{TM}}^n(l, m) h_{ln} X_{lm} - \frac{\mathrm{i}}{k_n} a_{\text{TE}}^n(l, m)\, \hat{*}\, \mathrm{d}(h_{ln} X_{lm}) \right) \mathrm{e}^{-\mathrm{i}k_n t} - \mathrm{d}\Phi_{\text{M}}^{\text{II}} \tag{3.23}$$

where $h_{ln} \equiv h_l^{(1)}(k_n r)$ and the vector harmonics

$$X_{lm} = \frac{r\#\, \mathrm{d}Y_{lm}}{i\sqrt{l(l+1)}} \tag{3.24}$$

satisfy

$$\int_{S_r^2} \bar{X}_{l'm'} \wedge \# X_{lm} = r^2 \delta_{l'l} \delta_{m'm} \tag{3.25}$$

$$\int_{S_r^2} \bar{X}_{l'm'} \wedge X_{lm} = 0 \qquad \forall l, l', m, m'. \tag{3.26}$$

(Here and below an overbar denotes complex conjugation.) The time-independent harmonic 0-forms may be expanded as

$$\Phi_{\text{E}}^{\text{II}} = \sum_{l=1}^{\infty} \frac{\alpha_{\text{E}}(l)}{r^{l+1}} P_l(\cos\theta) \tag{3.27}$$

$$\Phi_{\text{M}}^{\text{II}} = \sum_{l=1}^{\infty} \frac{\beta_{\text{M}}(l)}{r^{l+1}} P_l(\cos\theta) \tag{3.28}$$

in terms of the Legendre polynomials

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}. \tag{3.29}$$

Exclusion of $l=0$ incorporates the charge neutrality condition for the magnetised sphere and we observe that only odd/even values of l will contribute to the sum in equation (3.28)/(3.27). The vacuum Maxwell field F^{II} can now be computed in terms of the complex amplitudes $a^n_{\mathrm{TE}}(l,m)$, $a^n_{\mathrm{TM}}(l,m)$, $\alpha_{\mathrm{E}}(l)$, $\beta_{\mathrm{M}}(l)$ by imposing the standard boundary conditions:

$$[F^{\mathrm{I}}-F^{\mathrm{II}}]\wedge \mathrm{d}r|_{r=a}=0 \tag{3.30}$$

where, in terms of the interior electric and magnetic fields,

$$F^{\mathrm{I}}=-\boldsymbol{e}^{\mathrm{I}}\wedge \mathrm{d}t-\hat{*}\boldsymbol{b}^{\mathrm{I}}. \tag{3.31}$$

Since the time-dependent part of our interior fields depends on the combination $(\phi-\omega t)$, their Fourier expansions are

$$\boldsymbol{e}^{\mathrm{I}}=\sum_{n=-\infty}^{\infty}(\boldsymbol{e}^{\mathrm{I}}_n(r,\theta)\,\mathrm{e}^{\mathrm{i}n\phi})\,\mathrm{e}^{-\mathrm{i}k_n t}-\mathrm{d}\Phi^{\mathrm{I}}_{\mathrm{E}} \tag{3.32}$$

$$\boldsymbol{b}^{\mathrm{I}}=\sum_{n=-\infty}^{\infty}(\boldsymbol{b}^{\mathrm{I}}_n(r,\theta)\,\mathrm{e}^{\mathrm{i}n\phi})\,\mathrm{e}^{-\mathrm{i}k_n t}-\mathrm{d}\Phi^{\mathrm{I}}_{\mathrm{M}}. \tag{3.33}$$

Hence the only non-zero multipole amplitudes are given, with $m\neq 0$, by

$$a_{\mathrm{TE}}(l,m)\equiv a^m_{\mathrm{TE}}(l,m)=\frac{k_m a}{h^{(1)}_l(k_m a)\sqrt{l(l+1)}}\int_{S^2(a)}(\boldsymbol{b}^{\mathrm{I}}_m)_r\,\mathrm{e}^{\mathrm{i}m\phi}\,\bar{Y}_{lm}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{3.34}$$

$$a_{\mathrm{TM}}(l,m)\equiv a^m_{\mathrm{TM}}(l,m)=\frac{k_m}{\sqrt{l(l+1)}}[\partial_r(rh^{(1)}_l(k_m r))]^{-1}_{r=a}\int_{S^2(a)}\mathrm{d}\#[(\boldsymbol{e}^{\mathrm{I}}_m)_t\,\mathrm{e}^{\mathrm{i}m\phi}]\,\bar{Y}_{lm} \tag{3.35}$$

where the interior radial magnetic field mode is determined from

$$(\boldsymbol{b}^{\mathrm{I}}_m)_r=i_{\partial_r}\boldsymbol{b}^{\mathrm{I}}_m \tag{3.36}$$

and the interior tangential electric field mode follows from the decomposition

$$\boldsymbol{e}^{\mathrm{I}}_n=(\boldsymbol{e}^{\mathrm{I}}_n)_r\,\mathrm{d}r+(\boldsymbol{e}^{\mathrm{I}}_n)_t. \tag{3.37}$$

Thus for the particular magnetisations discussed in (2.27) and (2.50) the time-dependent radial and tangential fields are:

$$(\boldsymbol{b}^{\mathrm{I}_a})_r=(b_1\cos\alpha-b_2\sin\alpha)\sin\theta \tag{3.38}$$

$$(\boldsymbol{e}^{\mathrm{I}_a})_t=(-b_1\cos\alpha+b_2\sin\alpha)\omega r\sin^2\theta\,e^2 \tag{3.39}$$

$$(\boldsymbol{b}^{\mathrm{I}_b})_r=\gamma\mathscr{P}(\gamma)(b_1\cos\alpha-b_2\sin\alpha)\sin\theta \tag{3.40}$$

$$(\boldsymbol{e}^{\mathrm{I}_b})_t=\gamma\mathscr{P}(\gamma)(-b_1\cos\alpha+b_2\sin\alpha)\omega r\sin^2\theta\,e^2. \tag{3.41}$$

One finds for the non-zero amplitudes in the case of a pliable magnetisation:

$$a_{\mathrm{TE}}(l,m)=\tfrac{1}{2}(b_1-mib_2)\frac{k_m a}{h^{(1)}_l(k_m a)\sqrt{l(l+1)}}\int_{S^2(a)}\gamma\mathscr{P}(\gamma)\,\mathrm{e}^{\mathrm{i}m\phi}\,\bar{Y}_{lm}\sin^2\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{3.42}$$

$$a_{\mathrm{TM}}(l,m)=-\tfrac{1}{2}(b_1-mib_2)\frac{k_m\omega a^2}{\sqrt{l(l+1)}}[\partial_r(rh^{(1)}_l(k_m r))]^{-1}_{r=a}$$
$$\times\int_{S^2(a)}\partial_\theta[\gamma\mathscr{P}(\gamma)\sin^3\theta]\,\mathrm{e}^{\mathrm{i}m\phi}\,\bar{Y}_{lm}\,\mathrm{d}\theta\wedge\mathrm{d}\phi \tag{3.43}$$

where $m = 1, -1$ and

$$\alpha_{\mathrm{E}}(l) = \frac{2l+1}{2} a^{l+1} \int_{-1}^{1} \Phi^{\mathrm{l}}_{\mathrm{E}}(a, \theta) P_l(\mu)\, \mathrm{d}\mu \tag{3.44}$$

$$\beta_{\mathrm{M}}(l) = \frac{2l+1}{2} \frac{a^{l+1}}{l+1} \int_{-1}^{1} b^{\mathrm{l}}_r(a, \theta) P_l(\mu)\, \mathrm{d}\mu \tag{3.45}$$

where

$$\Phi^{\mathrm{l}}_{\mathrm{E}}(r, \theta) = \frac{b_3}{\omega} \ln \gamma \tag{3.46}$$

$$b^{\mathrm{l}}_r(r, \theta) = b_3 \cos\theta\, \gamma^2 \tag{3.47}$$

$$\mu \equiv \cos\theta. \tag{3.48}$$

Thus the exterior solution is represented as a multiple series in terms of the constants $\{b_i\}$ (to be determined) and the parameters ω, a and $\{m_i\}$.

In the case of the stiff magnetisation it follows from (3.38), (3.39) that the interior Fourier components are:

$$(b^{\mathrm{I}\omega}_{1})_r = \tfrac{1}{2}(b_1 - \mathrm{i}b_2) \sin\theta \tag{3.49}$$

$$(e^{\mathrm{I}\omega}_{1})_r = -\tfrac{1}{2}(b_1 - \mathrm{i}b_2)\omega a \sin^2\theta\, e^2 \tag{3.50}$$

$$(b^{\mathrm{I}\omega}_{-1})_r = \tfrac{1}{2}(b_1 + \mathrm{i}b_2) \sin\theta \tag{3.51}$$

$$(e^{\mathrm{I}\omega}_{-1})_r = -\tfrac{1}{2}(b_1 + \mathrm{i}b_2)\omega a \sin^2\theta\, e^2. \tag{3.52}$$

Hence the only non-zero amplitudes are

$$a_{\mathrm{TE}}(1, 1) = -(b_1 - \mathrm{i}b_2) \sqrt{\frac{\pi}{3}} \frac{\omega a}{h^{(1)}_1(\omega a)} \tag{3.53}$$

$$a_{\mathrm{TE}}(1, -1) = -(b_1 + \mathrm{i}b_2) \sqrt{\frac{\pi}{3}} \frac{\omega a}{h^{(1)}_1(-\omega a)} \tag{3.54}$$

$$a_{\mathrm{TM}}(2, 1) = (b_1 - \mathrm{i}b_2) \sqrt{\frac{\pi}{5}} \frac{\omega^2 a^2}{[\partial_r(r h^{(1)}_2(\omega r))]_{r=a}} \tag{3.55}$$

$$a_{\mathrm{TM}}(2, -1) = (b_1 + \mathrm{i}b_2) \sqrt{\frac{\pi}{5}} \frac{\omega^2 a^2}{[\partial_r(r h^{(1)}_2(-\omega r))]_{r=a}}. \tag{3.56}$$

4. The discontinuity current

To fix the constants $\{b_i\}$ in terms of the magnetisation we now compute the surface 4-current and demand that the surface of the sphere in an inertial frame carry no surface 3-current when $\omega = 0$. The discontinuity 4-current j_Δ, a 1-form on the hypersurface $r = a$, is given by the boundary condition

$$i_{\partial_r}[G^{\mathrm{I}} - G^{\mathrm{II}}]|_{r=a} = -j_\Delta \tag{4.1}$$

where $G^{\mathrm{I}} = \varepsilon_0 F^{\mathrm{I}} + \Pi^{\mathrm{I}}$ and

$$G^{\mathrm{II}} = \varepsilon_0 F^{\mathrm{II}} = \varepsilon_0(-e^{\mathrm{II}} \wedge e^0 - \hat{*} b^{\mathrm{II}}). \tag{4.2}$$

The inertial surface charge density 2-form Σ_U and inertial surface 3-current density 1-form $\mathscr{K}_U$ are given in terms of j_Δ by

$$*j_\Delta = \Sigma_U \wedge \mathrm{d}f + \mathscr{K}_U \wedge \mathrm{d}f \wedge \tilde{U} \tag{4.3}$$

where $i_U \Sigma_U = i_U \mathscr{K}_U = 0$ and $f = r - a$. Hence for the pliable magnetisation case we compute from (2.52), (3.1), (3.22), (3.23) and (4.1)-(4.3):

$$\Sigma^b_{\partial_t} = \bigg[c_3\gamma^2\omega r \sin^2\theta - \varepsilon_0\partial_r\Phi^{\mathrm{II}}_{\mathrm{E}} + \gamma(\mathscr{H}_1(\gamma)\cos\alpha - \mathscr{H}_2(\gamma)\sin\alpha)\omega r \sin\theta\cos\theta$$
$$- \sum_{l,m} \varepsilon_0 \frac{\sqrt{l(l+1)}}{\omega r} a_{\mathrm{TM}}(l,m) h_{lm} Y_{l,m}\, \mathrm{e}^{-\mathrm{i}k_m t} \bigg] e^2 \wedge e^3 \bigg|_{r=a} \tag{4.4}$$

$$\mathscr{K}^b_{\partial_t} = \bigg[-c_3\gamma^2 \sin\theta\, e^2 + \varepsilon_0\left(\frac{1}{r}\partial_\theta \Phi^{\mathrm{II}}_{\mathrm{M}}\right) e^2 + \gamma(\mathscr{H}_1(\gamma)\cos\alpha - \mathscr{H}_2(\gamma)\sin\alpha)\cos\theta\, e^2$$
$$+ \gamma(\mathscr{H}_1(\gamma)\sin\alpha + \mathscr{H}_2(\gamma)\cos\alpha) e^3$$
$$+ \sum_{l,m} \varepsilon_0 \frac{\mathrm{i}}{\sqrt{l(l+1)}} a_{\mathrm{TM}}(l,m) r h_{lm} \# \,\mathrm{d} Y_{lm}\, \mathrm{e}^{-\mathrm{i}k_m t}$$
$$- \sum_{l,m} \varepsilon_0 \frac{1}{\omega\sqrt{l(l+1)}} a_{\mathrm{TM}}(l,m) \partial_r (r h_{lm})\, \mathrm{d} Y_{lm}\, \mathrm{e}^{-\mathrm{i}k_m t} \bigg]_{r=a}. \tag{4.5}$$

One may verify that the total inertial surface charge $Q^b_{\mathrm{S}} = \int_{S^2_a} \Sigma^b_{\partial_t}$ is equal and opposite to the total volume charge $Q^b_{\mathrm{vol}} = \int_{S^2_a} *G^{\mathrm{I}b}$. Furthermore Q^b_{S} approaches zero as ω tends to zero. In such a limit ($\gamma \to 1$) we note that $\mathscr{K}^b_{\partial_t}$ vanishes when we choose

$$b_i = \tfrac{2}{3}\mu_0 m_i \qquad i = 1, 2, 3. \tag{4.6}$$

The computation for the stiff magnetisation case proceeds in a similar manner but is simpler. We find

$$\Sigma^a_{\partial_t} = \bigg[c_3\gamma^2\omega r \sin^2\theta - \varepsilon_0\partial_r\Phi^{\mathrm{II}}_{\mathrm{E}} + (c_1\cos\alpha - c_2\sin\alpha)\omega r\sin\theta\cos\theta$$
$$- \sum_{m=1,-1} \varepsilon_0 \frac{\sqrt{6}}{\omega r} a_{\mathrm{TM}}(2,m) h^{(1)}_2(k_m a) Y_{2m}\, \mathrm{e}^{-\mathrm{i}k_m t} \bigg] e^2 \wedge e^3 \bigg|_{r=a} \tag{4.7}$$

$$\mathscr{K}^a_{\partial_t} = \bigg[-c_3\gamma^2\sin\theta\, e^2 + \varepsilon_0\left(\frac{1}{r}\partial_\theta\Phi^{\mathrm{II}}_{\mathrm{M}}\right) e^2 + (c_1\cos\alpha - c_2\sin\alpha)\cos\theta\, e^2$$
$$+ (c_1\sin\alpha + c_2\cos\alpha) e^3$$
$$+ \sum_{m=1,-1} \varepsilon_0 \frac{\mathrm{i}}{\sqrt{6}} a_{\mathrm{TM}}(2,m) r h^{(1)}_2(k_m a) \# \mathrm{d} Y_{2m}\, \mathrm{e}^{-\mathrm{i}k_m t}$$
$$- \sum_{m=1,-1} \varepsilon_0 \frac{1}{\omega\sqrt{2}} a_{\mathrm{TE}}(1,m) \partial_r (r h^{(1)}_1(k_m r))\, \mathrm{d} Y_{1m}\, \mathrm{e}^{-\mathrm{i}k_m t} \bigg]_{r=a} \tag{4.8}$$

and again note the equality between Q^a_{S} and Q^a_{vol}. Furthermore the condition $\mathscr{K}^a_{\partial_t}|_{\omega=0} = 0$ leads to the same relation between the constants b_i and m_i, namely equation (4.6). Thus (2.27), (2.50) completely specifies the interior and exterior fields in terms of the magnetisation strength (m_1, m_2, m_3) and the parameters ω and a.

5. The torque reaction

The detailed structure of the multipole series for case b is determined by the functions $\mathscr{P}(\gamma)$ and $\mathscr{H}(\gamma)$ that control the size of the various multipole amplitudes. Such functions are monotonic in the range $[1, \infty)$ with bounded variation. One may expand the multipole amplitudes as a power series in ωa to any desired degree of accuracy. There is certainly a range of ωa for which higher-order multipoles can play a non-trivial role in the details of the radiation pattern. Such modifications to the simple dipole model may be compared with alternative models (such as the off-centre point dipole [13]) that may produce qualitatively similar effects. As we noted in [10] the inertial surface charge and 3-current densities have a significant dependence on the presence of the high-order unipolar induction harmonics.

To gain further insight into our solutions we have examined the torque experienced by a stiffly magnetised sphere due to the radiation of angular momentum. This may be contrasted with the average radiated power $\langle U \rangle$. Such a torque is generated by the angular momentum current that depends on the time-dependent components of F^{II}. In general the inertial electric and magnetic fields defiened in (3.19), (3.20) have the Fourier expansions:

$$e = \sum_{n=-\infty}^{\infty} e_n(r, \theta, \phi)\, \mathrm{e}^{-\mathrm{i}k_n t} \tag{5.1}$$

$$b = \sum_{n=-\infty}^{\infty} b_n(r, \theta, \phi)\, \mathrm{e}^{-\mathrm{i}k_n t} \tag{5.2}$$

where

$$e_n = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \left(\frac{\mathrm{i}}{k_n} a^n_{\mathrm{TM}}(l, m) \,\hat{*}\, \mathrm{d}(h_{ln} X_{lm}) + a^n_{\mathrm{TE}}(l, m) h_{ln} X_{lm} \right) \tag{5.3}$$

$$b_n = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \left(a^n_{\mathrm{TM}}(l, m) h_{ln} X_{lm} - \frac{\mathrm{i}}{k_n} a^n_{\mathrm{TE}}(l, m) \,\hat{*}\, \mathrm{d}(h_{ln} X_{lm}) \right). \tag{5.4}$$

The time-averaged torque may be computed from the contribution of the radiative part of F^{II} to the field angular momentum current. Thus it is convenient to introduce the 2-form $\mathscr{F}$:

$$\mathscr{F} = \mathrm{d}t \wedge e - \hat{*} b \tag{5.5}$$

$$= \sum_{n=-\infty}^{\infty} (\mathrm{d}t \wedge e_n - \hat{*} b_n)\, \mathrm{e}^{-\mathrm{i}k_n t} \tag{5.6}$$

$$= \sum_{n=-\infty}^{\infty} \mathscr{F}_n\, \mathrm{e}^{-\mathrm{i}k_n t} \tag{5.7}$$

where the Fourier components satisfy the reality condition

$$\bar{\mathscr{F}}_n = \mathscr{F}_{-n}. \tag{5.8}$$

In terms of the Killing vector ∂_ϕ that generates rotations about the axis of rotation of the magnetised sphere we define the radiative angular momentum current 3-form as

$$\tau_{\partial_\phi} = \tfrac{1}{2}[\, i_{\partial_\phi} \mathscr{F} \wedge * \mathscr{F} - i_{\partial_\phi} * \mathscr{F} \wedge \mathscr{F}]. \tag{5.9}$$

The time-averaged 3-form becomes

$$\langle \tau_{\partial_\phi} \rangle = \frac{1}{2} \sum_{n=-\infty}^{\infty} [\, i_{\partial_\phi} \mathscr{F}_n \wedge * \bar{\mathscr{F}}_n - i_{\partial_\phi} * \bar{\mathscr{F}}_n \wedge \mathscr{F}_n]. \tag{5.10}$$

To calculate the average torque on the rotating sphere we compute the (time-averaged) rate of angular momentum flux crossing a sphere of radius $r > a$. Since the total angular momentum 3-form is closed in II we may evaluate our results for a value of r that permits us to use the asymptotic expression for the spherical Hankel functions. The average torque in the inertial frame is defined as

$$\langle \tau \rangle = -\int_{S^2(r)} i_{\partial_t} \langle \tau_{\partial_\phi} \rangle \tag{5.11}$$

$$= -\sum_{n=-\infty}^{\infty} \int_{S^2(r)} [(i_{\partial_\phi} \boldsymbol{e}_n) \,\hat{*}\, \bar{\boldsymbol{e}}_n + (i_{\partial_\phi} \bar{\boldsymbol{b}}_n) \,\hat{*}\, \boldsymbol{b}_n]. \tag{5.12}$$

Then using (4.3) we have

$$\int_{S^2(r)} (i_{\partial_\phi} \boldsymbol{e}_n) \,\hat{*}\, \bar{\boldsymbol{e}}_n$$

$$= \int_{S^2(r)} \sum_{l,m} \sum_{l',m'} \Bigg[\left(\frac{1}{k_n^2 r} \right) \sqrt{\frac{l'(l'+1)}{l(l+1)}}$$

$$\times a^n_{\mathrm{TM}}(l, m) \bar{a}^n_{\mathrm{TM}}(l', m') \frac{\mathrm{d}}{\mathrm{d}r} (r h_{ln}) \bar{h}_{l'n} \partial_\phi Y_{lm} \bar{Y}_{l'm'} e^2 \wedge e^3 \Bigg]$$

$$+ \int_{S^2(r)} \sum_{l,m} \sum_{l',m'} \Bigg[\left(\frac{\mathrm{i}}{k_n} \right) \sqrt{\frac{l'(l'+1)}{l(l+1)}}$$

$$\times a^n_{\mathrm{TE}}(l, m) \bar{a}^n_{\mathrm{TM}}(l', m') h_{ln} \bar{h}_{l'n} \sin\theta \, \partial_\theta Y_{lm} \bar{Y}_{l'm'} e^2 \wedge e^3 \Bigg]. \tag{5.13}$$

Similarly from (5.4)

$$\int_{S^2(r)} (i_{\partial_\phi} \bar{\boldsymbol{b}}_n) \,\hat{*}\, \boldsymbol{b}_n$$

$$= \int_{S^2(r)} \sum_{l,m} \sum_{l',m'} \Bigg[\left(\frac{1}{k_n^2 r} \right) \sqrt{\frac{l(l+1)}{l'(l'+1)}}$$

$$\times a^n_{\mathrm{TE}}(l, m) \bar{a}^n_{\mathrm{TE}}(l', m') \frac{\mathrm{d}}{\mathrm{d}r} (r \bar{h}_{l'n}) h_{ln} \partial_\phi \bar{Y}_{l'm'} Y_{lm} e^2 \wedge e^3 \Bigg]$$

$$+ \int_{S^2(r)} \sum_{l,m} \sum_{l',m'} \Bigg[\left(\frac{\mathrm{i}}{k_n} \right) \sqrt{\frac{l(l+1)}{l'(l'+1)}}$$

$$\times a^n_{\mathrm{TE}}(l, m) \bar{a}^n_{\mathrm{TM}}(l', m') h_{ln} \bar{h}_{l'n} \sin\theta \, \partial_\theta \bar{Y}_{l'm'} Y_{lm} e^2 \wedge e^3 \Bigg]. \tag{5.14}$$

The interference terms involving the products $a_{\mathrm{TM}} a_{\mathrm{TE}}$ in these expressions cancel in the average torque. One finds that they contribute

$$\langle \tau_X \rangle = \sum_{n=-\infty}^{\infty} \sum_{l,l',m} S^n(l, l', m) \left\{ \sqrt{\frac{l'(l'+1)}{l(l+1)}} \, I^m_{ll'} + \sqrt{\frac{l(l+1)}{l'(l'+1)}} \, I^m_{l'l} \right\} \tag{5.15}$$

where

$$S^n(l, l', m) = \frac{\mathrm{i}r^2}{k_n} A_{lm}A_{l'm}h_{ln}\bar{h}_{l'n}a^n_{\mathrm{TE}}(l, m)\bar{a}^n_{\mathrm{TM}}(l', m) \tag{5.16}$$

$$I^m_{ll'} = \int_{-1}^{1} (x^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}x} P^m_l(x)\right) P^m_{l'}(x)\,\mathrm{d}x \tag{5.17}$$

$$Y_{lm} = A_{lm}P^m_l(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}. \tag{5.18}$$

From the properties of the associated Legendre functions we observe that for $|l'-l| \neq 1$, $I_{ll'} = 0$. Furthermore it follows that the bracketed term in (5.15) vanishes for $|l'-l| = 1$. From (5.13), (5.14) we have

$$\langle\tau\rangle = -\sum_{n=-\infty}^{\infty}\sum_{l,m}\left(\frac{\mathrm{i}mr}{k_n^2}\right)[\bar{h}_{ln}\partial_r(rh_{ln})|a^n_{\mathrm{TM}}(l, m)|^2 + h_{ln}\partial_r(r\bar{h}_{ln})|a^n_{\mathrm{TE}}(l, m)|^2]. \tag{5.19}$$

Using $h_l^{(1)}(k_nr) \to (-\mathrm{i})^{l+1}\,\mathrm{e}^{\mathrm{i}k_nr}/k_nr$ for r in the radiation zone

$$\langle\tau\rangle = \sum_{n=-\infty}^{\infty}\sum_{l,m}\frac{m}{k_n^3}[|a^n_{\mathrm{TM}}(l, m)|^2 + |a^n_{\mathrm{TE}}(l, m)|^2]. \tag{5.20}$$

This result is general. For our rigidly rotating solutions we recall that only amplitudes with $m = n \neq 0$ contribute. Hence

$$\langle\tau\rangle = \frac{1}{\omega^3}\sum_{l,m\neq 0}\frac{1}{m^2}[|a_{\mathrm{TM}}(l, m)|^2 + |a_{\mathrm{TE}}(l, m)|^2]. \tag{5.21}$$

One may evaluate the average power radiated in a similar way by using the Killing vector ∂_t instead of ∂_ϕ in (4.9). One obtains in this manner the general time-averaged power:

$$\langle U\rangle = \sum_{n=-\infty}^{\infty}\sum_{l,m}\frac{1}{k_n^2}[|a^n_{\mathrm{TM}}(l, m)|^2 + |a^n_{\mathrm{TE}}(l, m)|^2] \tag{5.22}$$

in terms of the multipole amplitudes. Hence we see that for our rigidly rotating sphere

$$\langle U\rangle = \frac{1}{\omega^2}\sum_{l,m\neq 0}\frac{1}{m^2}[|a_{\mathrm{TM}}(l, m)|^2 + |a_{\mathrm{TE}}(l, m)|^2] \tag{5.23}$$

so that

$$\omega\langle\tau\rangle = \langle U\rangle. \tag{5.24}$$

Using the explicit form of the amplitudes in (3.53)-(3.56) for the stiff magnetisation case we find that the time averaged torque is

$$\langle\tau\rangle = \left(\frac{8\pi}{9}\right)\mu_0^2(m_1^2 + m_2^2)\left\{\frac{\omega^3a^6}{3(1+\omega^2a^2)} + \frac{\omega^7a^{10}}{5(36-3\omega^4a^4+\omega^6a^6)}\right\}. \tag{5.25}$$

In the ultrarelativistic limit as $\omega a \to 1$

$$\langle\tau\rangle \to \left(\frac{352\pi a^3}{2295}\right)\mu_0^2(m_1^2 + m_2^2).$$

It is customary to estimate a pulsar spin-down rate by assuming that the radiated energy accounts for most of the loss of rotational kinetic energy. Strictly speaking one should not rely on (5.23) for a pulsar with a rapidly changing ω since the derivation has taken the angular speed constant throughout. However, if one assumes that it is reasonable to assert

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}I\omega^2(t)\right) = -\langle U(\omega)\rangle \tag{5.26}$$

(which is certainly consistent with $\mathrm{d}/\mathrm{d}t(I\omega) = -\langle\tau(\omega)\rangle$) for some constant moment of inertia I, one may use the radiated power formula to estimate the change of pulsar period. For models based on magnetic dipole radiation one obtains, for some constant κ:

$$\dot{\omega} = -\kappa\omega^{n_b} \tag{5.27}$$

with a breaking index $n_b = 3$. This is also the equation determined by (5.25) for $\omega a \ll 1$. However for a general ω the notion of a single breaking index is not applicable and one gets a different kind of differential equation for $\omega(t)$ from (4.25).

Although the particular magnetisations discussed in this paper may have little detailed relevance for magnetic stellar interiors, the general features inherent in higher-order multipole effects may be worthy of further consideration in astrophysical contexts. The high ω dependence of the average torque is clearly a property not shared by magnetic dipole models. It would be of interest to see how the presence of relativistic multipole emissions affects the evolution of a pulsar in a more realistic model, based for instance on a magnetised sphere rotating in a magneto-plasma. In this context we note that the recently observed (optical) pulsar from the 1987A supernova is estimated to have an equatorial speed one half that of light.

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